The derivative of a function of Complex Variable:

Definition1.4. Let $w = f(z)$ be a single valued function defined in a domain D of the Complex plane and z_0 be any fixed point in D. Then f(z) is said to be differentiable at a point $z_0 \in D$ if the increment ratio

$$\frac{f(z)-f(z_0)}{z-z_0}$$

tends to a unique finite limit as z tends to z_0 in any manner (i.e., z tends to z_0 along any path connecting z and z_0, lying entirely in D). In this case, this unique finite limit is called the derivative of f(z) at z = z_0 and is denoted by $f'(z_0)$. Thus $f'(z_0) = \frac{df(z_0)}{dz} = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$.

The above definition asserts that given any $\varepsilon > 0$, there exists a number $\delta > 0$, depending on ε and z_0, such that $\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon$ for all $z \in D'(z_0,\delta)$, where $D'(z_0,\delta) = \{z \in D : 0 < |z - z_0| < \delta\}$.

Note1.5. Writing $z - z_0 = h$ in the above definition we get the equivalent form

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

Replacing z_0 by any given point $z \in D$, we have

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

If instead of h, we write Δz, we have,

$$f'(z) = \frac{df(z)}{dz} = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

These equivalent forms are used in solving different problems.

Note1.6. If we get different values of this limit as $z \to z_0$ from different points of D, i.e., if the limit depends upon $amp\Delta z$, we say that the derivative of f(z) at z = z_0 does not exist and the function f(z) is said to be non-differentiable at z = z_0. Hence, in order to prove the non-differentiability of f(z), we should try different paths for Δz. Convenient paths for Δz are along real and imaginary axes, i.e., by taking Δz either wholly real or wholly imaginary.

<u>**Definition1.5. (Analytic at a point)**</u> A single valued function f is said to be **analytic at a point z_0,** if there exists some $\delta-$neighbourhood of z_0 at all points of which $f'(z)$ exists.

<u>**Definition1.6. (Analytic in a region R).**</u> If a single valued function is analytic at every point z' of a region R, i.e., if for each $z' \in R$, there exists some $\delta-$neighbourhood of z' at all points of which $f'(z)$ exists., then f is said to be analytic in R.

<u>**Note1.7.**</u> Since a domain is an open connected set in the complex plane, we can say that a single valued function f is analytic in a domain D if it has a derivative at every point of D

<u>**Note1.8.**</u> The terms regular and holomorphic are used as synonyms for analytic.

Some authors, however, prefer to make distinction between the terms analytic and regular in the following sense.

If a function f(z) is single valued and differentiable at every point of its domain D, except possibly for a finite number of exceptional points, called singular points, then f(z) is analytic in D. If, however, no point of D is a singularity, then we say that f(z) is regular in D.

Note1.9.* We would like to mention here that a function which is differentiable at a point, need not necessarily be analytic at that point. To establish this remark, we shall consider the exercise1.7.

Exercise1.7. Give an example of a function which is not analytic a point z_0 in the complex plane although $f'(z_0)$ exists. **(2 marks)**

Ans. We consider the function $f(z) = |z|^2$ for all $z \in C$. Then for $z_0 \in C,\ z_0 \neq 0$, we have

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{|z|^2-|z_0|^2}{z-z_0} = \frac{z.\bar{z}-z_0\overline{z_0}}{z-z_0} = \frac{\bar{z}(z-z_0)+z_0(\bar{z}-\overline{z_0})}{z-z_0} = \bar{z}+z_0\frac{\bar{z}-\overline{z_0}}{z-z_0} = \bar{z}+z_0\frac{\overline{z-z_0}}{z-z_0} \quad \text{............(i)}$$

Let (z-z₀) = $re^{i\theta}$, r>0. Then $\overline{z-z_0} = re^{-i\theta}$. So $\frac{\overline{z-z_0}}{z-z_0} = e^{-2i\theta} = (\cos 2\theta - i\sin 2\theta)$. Therefore, by (i) we have $\frac{f(z)-f(z_0)}{z-z_0} = \bar{z}+z_0(\cos 2\theta - i\sin 2\theta)$, which obviously depends on $\theta = \arg(z-z_0)$. Hence if $z_0 \neq 0$, then $\frac{f(z)-f(z_0)}{z-z_0}$ does not tend to a unique limit as $z \to z_0$ in any manner.

So, $f'(z)$ does not exist at any z = $z_0 \neq 0$.

But when $z_0 = 0$, we have by (i), $\frac{f(z)-f(z_0)}{z-z_0} = \bar{z}$, which tends to 0 as z approaches 0 in any manner. Hence, $f'(0)$ exists.

Note1.9. A simple answer of the above problem, using Cauchy Riemann Equations, will be given after the proof of C-R equations.

Necessary condition for a function f(z) to be analytic

Theorem.(Cauchy Riemann Partial Differential Equations). A necessary condition for a function $f(z) = u(x,y)+iv(x,y)$, defined in a domain D, to be differentiable at a point z =x + i y is that the four partial derivatives u_x, v_x, u_y, v_y should exist and satisfy the Cauchy-Riemann differential equations

Proof. Let $f(z) = u(x,y)+iv(x,y)$, be differentiable at a given point z = x + i y. Then the increment ratio $\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{u(x+\Delta x, y+\Delta y)-u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y)-v(x,y)}{\Delta x+i\Delta y}$ must tend to a definite limit $f'(z)$ as $\Delta z \to 0$ in any manner.

Now if $\Delta z = \Delta x + i\Delta y$ approaches 0 along the real axis, then $\Delta y = 0$. So $\Delta z = \Delta x$. Therefore ,

$$f'(z) = \lim_{\Delta z \to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}\right] \quad \text{..................(i)}$$

Since the limit in the L. H. S. of (i) exists, the individual limits of the real and imaginary parts on the R.H.S. of (i) also must exist. So, $u_x(x,y)$ *and* $v_x(x,y)$ both exist and we have

$$f'(z) = u_x(x,y)+iv_x(x,y) \quad \text{.........................(ii)}$$

Again if Δz approaches 0 along the imaginary axis, then $\Delta x = 0$, so that $\Delta z = i\Delta y$. Therefore,

$$f'(z) = \lim_{\Delta y \to 0}\left[\frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y} + i\frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y}\right].$$

$$= \lim_{\Delta y \to 0}\left[\frac{v(x,y+\Delta y)-v(x,y)}{\Delta y} - i\frac{u(x,y+\Delta y)-u(x,y)}{\Delta y}\right]$$

Applying the similar reasoning as given in case of first approach of Δz towards 0, we see that both $u_y(x,y)$ *and* $v_y(x,y)$ exist and

$$f'(z) = v_y(x,y) - iu_y(x,y) \quad \text{..............................(iii)}$$

From (ii) and (iii) we see that all the four partial derivatives u_x, v_x, u_y, v_y exist and are connected by

$$u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$$

Equating the real and imaginary parts, we get $u_x = v_y$ and $v_x = -u_y$ i.e., $u_x = v_y$ and $u_y = -v_x$.

Note1.10. That the conditions of the above theorem are not sufficient can be shown through the following Examples.

Exercise1.8.*** [BU(H)2016] Show that function $f(z) = |xy|^{\frac{1}{2}}$, where z = x + iy, is not analytic at the origin, although the Cauchy-Riemann equations are satisfied at that point.

<u>Ans</u>. Let f(z) = u (x, y) +i v(x , y) so that $u(x,y) = |xy|^{\frac{1}{2}}$ and v (x ,y) = 0. So, $u_x(0,0) = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$, $u_y(0,0) = \lim_{k\to 0} \frac{u(0,0+k) - u(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$

$v_x(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0$, $v_y(0,0) = \lim_{k\to 0} \frac{0-0}{k} = 0$ .Thus $u_x(0,0) = v_y(0,0)$ and $u_y(0,0) = -v_x(0,0)$.

So, Cauchy-Riemann equations are satisfied for the given function f(z) at the origin.

$$\text{Now, } \lim_{z\to 0} \frac{f(0+z) - f(0)}{z} = \lim_{z\to 0} \frac{|xy|^{\frac{1}{2}} - 0}{x+iy}$$

Now, if $z \to 0$ along the line y = mx , where m is a real parameter.

$$\lim_{z\to 0} \frac{f(0+z) - f(0)}{z} = \lim_{x\to 0} \frac{|mx^2|^{\frac{1}{2}} - 0}{x+imx} = \lim_{x\to 0} \frac{|m|^{\frac{1}{2}}}{1+im} = \frac{|m|^{\frac{1}{2}}}{1+im}$$

Thus the limit, $\lim_{z\to 0} \frac{f(0+z) - f(0)}{z}$ depends on the parameter m and so the limit is not unique. Hence $f'(0)$ does not exists and therefore f is not analytic at z = 0.

Exercise1.9.*** [BU(H) 2007]. Show that the function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, z = x +i y $(\neq 0)$

=0 if z = 0,

satisfies the Cauchy-Riemann equation at z = 0. Does $f'(0)$ exist? Answer with justification.

<u>Ans.</u> Here $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ =u(x ,y) + i v(x , y) (say). Then $u(x,y) = \frac{x^3 - y^3}{x^2+y^2}$ and $v(x,y) = \frac{x^3 + y^3}{x^2+y^2}$.

$$\text{So, } u_x(0,0) = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1,$$

$$u_y(0,0) = \lim_{k\to 0} \frac{u(0,0+k) - u(0,0)}{k} = \lim_{k\to 0} \frac{-\frac{k^3}{k^2} - 0}{k} = -1$$

$$v_x(0,0)=\lim_{h\to 0}\frac{v(0+h,0)-v(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h^3}{h^2}-0}{h}=1$$ and

$$v_y(0,0)=\lim_{k\to 0}\frac{v(0,0+k)-v(0,0)}{k}=\lim_{k\to 0}\frac{\frac{k^3}{k^2}}{k}=1$$

Therefore, $u_x=v_y$ and $u_y=-v_x$ so that the Cauchy-Riemann equations are satisfied.

Now $$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{z\to 0}\frac{\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}}{x+iy}$$

We first assume that $z\to 0$ along the real axis. Then y = 0 and so

$$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{x\to 0}\frac{\frac{(x^3-0)+i(x^3+0)}{x^2+0}}{x+0}=1+i$$

We next assume that $z\to 0$ along the line y = x. Then

$$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{x\to 0}\frac{\frac{(x^3-x^3)+i(x^3+x^3)}{x^2+x^2}}{x+ix}=\frac{\frac{0+2ix^3}{2x^2}}{x(1+i)}=\frac{i}{1+i}=\frac{i(1-i)}{2}=\frac{(1+i)}{2}$$

Since the limit has different values for different approaches of z towards 0, we conclude that $f'(0)$ does not exist.

Exercise1.10.* ** [BU(H)2010] Examine whether the function $f(z)=\sin|z|$ is differentiable at z = 0. **(2marks)**

Ans. $$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{z\to 0}\frac{\sin|z|-\sin 0}{z}=\lim_{z\to 0}\frac{\sin\left(\sqrt{x^2+y^2}\right)}{x+iy}$$

We now suppose that $z\to 0$ along the real axis, on which y = 0. Then

$$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{x\to 0}\frac{\sin\left(\sqrt{x^2}\right)}{x}=\lim_{x\to 0}\frac{\sin x}{x}=1$$

We next assume that $z\to 0$ along the imaginary axis, on which x = 0. Then

$$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{y\to 0}\frac{\sin\left(\sqrt{y^2}\right)}{iy}=\frac{1}{i}\lim_{x\to 0}\frac{\sin y}{y}=\frac{1}{i}=-i$$

Since the limit has different values for different approaches of z towards 0 , we conclude that f is not differentiable at z = 0.

Exercise1.11. [BU(H)2006] Let $f:C\to C$ be defined by

$$f(z)=\frac{(\bar{z})^2}{z}\quad for\quad z\neq 0$$
$$=0\qquad for\quad z=0$$

Show that Cauchy-Riemann equations are satisfied at z = 0.Is the function f is derivable at z = 0? (2+1 marks)

Ans. For $z\neq 0$, we have $f(z)=\frac{(x-iy)^2}{x+iy}=\frac{(x-iy)^2(x-iy)}{x^2+y^2}=\frac{(x-iy)^3}{x^2+y^2}=\frac{x^3-3x^2iy+3x(i^2y^2)+(-iy)^3}{x^2+y^2}$

$$=\frac{(x^3-3xy^2)}{x^2+y^2}+i\frac{(y^3-3x^2y)}{x^2+y^2}$$

So, $u(x,y)=\dfrac{(x^3-3xy^2)}{x^2+y^2}$ and $v(x,y)=\dfrac{(y^3-3x^2y)}{x^2+y^2}$.

$$u_x(0,0)=\lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h^3}{h^2}}{h}=1,$$

$$u_y(0,0)=\lim_{k\to 0}\frac{u(0,0+k)-u(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0$$

$$v_x(0,0)=\lim_{h\to 0}\frac{v(0+h,0)-v(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

$$v_y(0,0)=\lim_{k\to 0}\frac{v(0,0+k)-v(0,0)}{k}=\lim_{k\to 0}\frac{\frac{k^3}{k^2}-0}{k}=1$$

Therefore, $u_x=v_y$ and $u_y=-v_x$ so that the Cauchy-Riemann equations are satisfied.

Now, $\displaystyle\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{z\to 0}\frac{\frac{(\bar{z})^2}{z}-0}{z}=\lim_{z\to 0}\frac{(x-iy)^2}{(x+iy)^2}$

Let z approach 0 along the real axis, on which y = 0

$$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{x\to 0}\frac{(x-0)^2}{(x+0)^2}=1$$

Next, let z approach 0 along the line y = x. Then

$$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{x\to 0}\frac{(x-ix)^2}{(x+ix)^2}=\frac{(1-i)^2}{(1+i)^2}=\frac{-2i}{2i}=-1$$

Since the two limits are different for different approaches of z towards 0 , we conclude that f is not derivable at z = 0.

Exercise1.12. [BU(H)2012] (3 marks). If $f(z)=\sin(2z-\bar{z})$, then show that $f'(0)$ does not exist.

Proof. Here $f(z)=\sin(2z-\bar{z})=\sin(2x+2iy-(x-iy))=\sin(x+3iy)$.

So, $\displaystyle\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{z\to 0}\frac{f(z)-0}{z}=\lim_{z\to 0}\frac{\sin(x+3iy)}{x+iy}$.

Let $z\to 0$ along the real axis , on which y = 0. Then

$$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{x\to 0}\frac{\sin(x+3i0)}{x+i0}=\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Next, let z approaches 0 along the imaginary axis, on which x = 0. Then

$$\lim_{z\to 0}\frac{f(0+z)-f(0)}{z}=\lim_{y\to 0}\frac{\sin(0+3iy)}{0+iy}=\lim_{y\to 0}\frac{3.\sin(3iy)}{3iy}=3. \quad \left[\textit{Since } \lim_{z\to 0}\frac{\sin z}{z}=1 \textit{ holds in complex plane}\right]$$

Since the two limits are different for different approaches of z towards 0, we conclude that $f'(0)$ does not exist.