SECTION 8: Nuclear Models – The Liquid Drop

Theoretical models of the nucleus encounter two principal problems:

(1) There is no exact mathematical expression that accounts for the nuclear force, unlike the atomic case, for which the electromagnetic force is well-defined by Coulomb's Law, and

(2) There is no mathematical solution to the many-body problem, a limitation shared by both nuclear and atomic systems.

Fortunately, the power of the computer permits calculations that use sophisticated approximations to minimize these problems and provide increasingly accurate models for describing nuclear and atomic properties.

The starting point for theoretical models of the nucleus treats the problem from two divergent perspectives: At the macroscopic extreme is the **Liquid Drop Model**, which examines the global properties of nuclei, such as energetics, binding energies, sizes, shapes and nucleon distributions. This model assumes that **all nucleons are alike** (other than charge). In contrast the **Shell Model** is designed to account for the quantal properties of nuclei such as spins, quantum states, magnetic moments and magic numbers. The basic assumption of the Shell Model is that **all nucleons are different**, i.e. nucleons are fermions and must occupy different quantum states, as is the case for atoms.

The idealized goal of theoretical nuclear physics is to combine these two concepts into a Unified Model that will describe both the macroscopic and microscopic aspects of nuclear matter in a single comprehensive framework.

<u>Justification for the Liquid Drop Model</u>

The basic assumption of the Liquid Drop Model is that the nucleus is a charged, nonpolar liquid drop held together by the nuclear force. In the simplest case, the chemical analogy would be a droplet of composed of nonpolar molecules such as CCl_4 or isopentane held together by Vander Waal's attraction. For such systems, the following properties are observed:

- The attractive force is short-ranged; i.e. there is a relatively sharp boundary at the surface, similar to our earlier discussions of a uniform density sphere or Woods-Saxon nucleon distribution.
- The force is saturated; i.e. all nucleons in the bulk of the liquid are bound equally, independent of radius.
- The nucleus is incompressible in its ground state, which accounts for the nearly uniform density distribution (Fig. 7.3) and constant average binding energy (Fig. 2.1).
- Surface tension is created by the loss in binding for nucleons on the nuclear surface, an effect that leads to a spherical shape to minimize the surface energy.

There are, however, significant differences between a classical liquid drop and a nucleus which must be accounted for in the model. For example:

- The nucleus has a limited number of particles (<270) compared to chemical systems ($\sim 10^{23}$). The net result is that there is a much larger fraction of nucleons on the surface relative to those in the bulk for nuclei compared to chemical systems.
- The nucleus is a two-component system composed of neutrons and protons.
- Protons in the nucleus carry positive electric charge. Therefore, they lose energy due to mutual Coulomb repulsion. An analogous chemical case would be an hypothetical cluster of Xe and Xe^+ ions held together by much stronger than normal Van de Waal's forces.
- Microscopic properties such as shell structure are not included in the model.

The Model

Taking the above liquid drop considerations into account, one can then construct a semi-empirical model (half theory/half data) to account for the total nuclear binding energy (or mass, Eqs.2.2a and 2.5c), the most basic of nuclear properties. The most important terms include:

(1) Nuclear Attraction – E_v

The strong nuclear force is equal among all nucleons, whether they are neutrons or protons. For an infinitely large nucleus, one can write

$$TBE = \langle BE \rangle A,$$

or more generally

$$TBE = C_1 A = E_V \qquad \text{(Eq. 8.1)}$$

where C_1 is a constant that describes the nuclear force in infinite nuclear matter, to be derived from fits to experimental data. **E_v** is called the '**volume term'**, since A is proportional to the volume,,

$$\text{Volume} = 4\pi R^3/3 = 4\pi r_o^3 (A^{1/3})^3 = (4\pi r_o^3)A$$

This is the dominant attractive term. In idealized neutron stars (~ infinite A) it is the only major term, since the proton concentration is low. However, the nucleus is small and charged due to the protons, therefore the binding energy is lowered due to the following effects:

(2) Surface Tension – E_S

Due to the fact that the nucleons on the surface are not surrounded by other nucleons, as they are in the bulk, they are not bound as tightly, giving rise to a '**surface term'**. The surface energy loss **E_S** for a sphere with surface tension constant σ is given by

$$E_S = -\sigma \text{ x Area} = -4\pi\sigma R^2 = -4\pi\sigma r_o^2 A^{2/3} \quad \text{, or}$$

$$E_S = -C_2 A^{2/3} \qquad \text{(Eq. 8.2)}$$

Since there is a greater fraction of the total number of nucleons on the surface compared to the bulk as the mass number A decreases, light nuclei lose binding energy due to surface energy loss. This effect is apparent from examination of the average binding energy curves in Figs. 2.1 and 2.2.

(3) Electric Charge Repulsion: Coulomb Energy – E_C

Since protons carry positive electric charge, they mutually repel one another in the nucleus This effect produces an additional binding energy loss that requires inclusion of a **Coulomb term E_C**. According to Coulomb's Law, the electrostatic energy of a charged sphere is

$$E_C = -(3/5)Z^2e^2/R, \text{ or, substituting for R,}$$

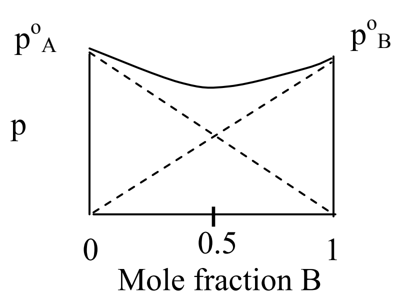
$$E_C = -C_3Z^2/A^{1/3} \qquad \text{(Eq. 8.3)}$$

The constant C_3 provides information relevant to nuclear sizes since it contains the radius parameter r_o .Due to the Z^2 dependence, nuclei with high atomic numbers lose binding energy, which accounts for the termination of the periodic chart when Z becomes large.

The surface energy, which affects light nuclei, and the Coulomb energy, which affects heavy nuclei, are the two major sources of binding energy loss in nuclei.

(4) Symmetry Energy – E_{sym}

According to Raoult's Law, in any two-component liquid with nonpolar attractive forces, the minimum in energy occurs when the two components occur in equal concentrations. The analogy to vapor pressure of the mixture is illustrated below.



That is, there is a minimum in the vapor pressure when there is an equal number of particles of each type, corresponding to maximum binding energy in the system. In nuclei this energy minimum is expressed in terms of the **Symmetry Energy, E_{sym}**, given by

$$E_{sym} = -C_4(N-Z)^2/A^2 \qquad \text{(Eq. 8.4)}$$

For nuclei with equal numbers of protons and neutrons, the '**symmetry term'** is zero, but whenever there is an excess of neutrons or protons, the nucleus pays a price in binding energy. The symmetry term for nuclei in their ground state serves as a base for understanding the environment in supernova explosions, where the neutron-to-proton ratio may diverge strongly from unity and the nuclear density may be unusually low or high.

(5) Pairing Energy – E_p

There are small but systematic differences in the total binding energies that depend on the even-odd character of the nucleus, giving rise to a term called the **Pairing Energy, E_p.** Nucleons of the same type prefer to occupy the same orbital and thus exist in pairs whenever possible (e.g. p-p and n-n). This is just the opposite of Hund's Rule in chemistry, where the electrons prefer separate orbits. How does one account for this behavior? It's the force. In atoms the electromagnetic repulsive force between electrons is minimized when the electrons are as far apart as possible, which favors different orbits. On the other hand, the nuclear force between nucleons is attractive. Therefore, in order to maximize the attraction, protons and neutrons pair up and share the same orbital whenever Z or N is even. This leave four possibilities as a function of mass number. If A is even, then one has either even-Z,even-N (e-e) or odd-Z,odd-N (o-o) nuclei. If A is odd, then the possibilities are even-Z,odd-N (e-o) or odd-Z,even-N (o-e). Evidence for the pairing effect is found in the number of stable isotopes of each type, as summarized in Table 8.1

Table 8.1

Number of stable isotopes as a function of even and odd proton and neutron numbers

	even Z even N (e-e)	even Z odd N (e-o)	odd Z even N (o-e)	odd Z odd N (o-o)
Number	157	55	50	4 (^{2}H,^{6}Li,^{10}B,^{14}N)

To account for the pairing energy, a fifth term is added to the binding energy equation,:

$$E_p = C_5\delta/A^{1/2}\text{ , where }\delta = \begin{cases} +1 & \text{e-e} \\ 0 & \text{e-o \& o-e} \\ -1 & \text{o-o} \end{cases} \qquad (\text{Eq. 5.5})$$

This term is defined so that even Z-even N nuclei gain binding energy, odd-A nuclei are not affected and odd Z-odd N nuclei lose binding energy.

Combining all five terms yields the **Liquid Drop Mass (or Binding Energy) Equation**, which is the basic form of the equation of state for nuclear matter:

$$\text{TBE} = C_1A - C_2A^{2/3} - C_3Z^2/A^{1/3} - C_4(N\text{-}Z)^2/A^2 + C_5\delta/A^{1/2} \qquad \text{Eq. 8.6a}$$

$$\langle BE\rangle = \underbrace{C_1}_{\text{Nuclear Attraction}} - \underbrace{C_2A^{-1/3}}_{\text{Surface Loss}} - \underbrace{C_3Z^2/A^{4/3}}_{\text{Charge Loss}} - \underbrace{C_4(N\text{-}Z)^2/A^3}_{\text{Asymmetry Loss}} + \underbrace{C_5\delta/A^{3/2}}_{\text{Pairing Gain/Loss}} \qquad \text{Eq. 8.6b}$$

The relative contribution of each of the terms is shown in Fig 8.1.

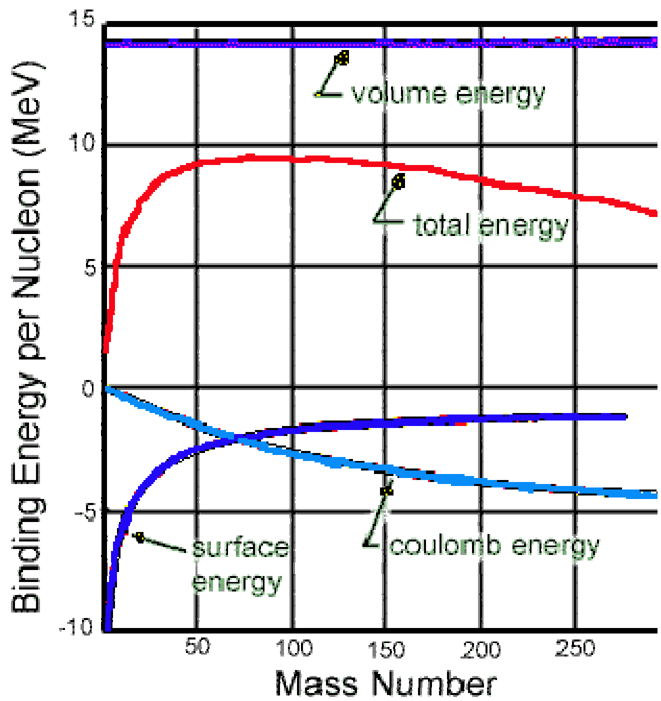


Fig. 8.1 Contribution of the volume, surface and Coulomb energies to the total average binding energy as a function of mass number.

By fitting the five terms in Eqs. 5.6 to known nuclear masses, the constants C_i can be derived, each of which has physical significance. The overall fit to the ~3000 known total nuclear binding energies is remarkably good to first order: ~ 5MeV or 0.005u. By adding additional terms, the fit can be improved. For example, the diffuse nuclear surface can be accounted for with a term

$$E_{diff} = - C_6\, Z^2/A.$$

Semiempirical mass equations have been developed with up to 250 parameters that predict total binding energies to within +/- 100 keV or 0.0001u.

In Fig8.2 the deviation between the liquid drop equation masses and measured values is plotted. While the fit is generally good, periodic strong deviations are observed at certain proton or neutron numbers. These deviations indicate additional stability and provide evidence for shell closures at nucleon numbers 2, 8, 20, 28, 50, 82 and 126 (neutrons).

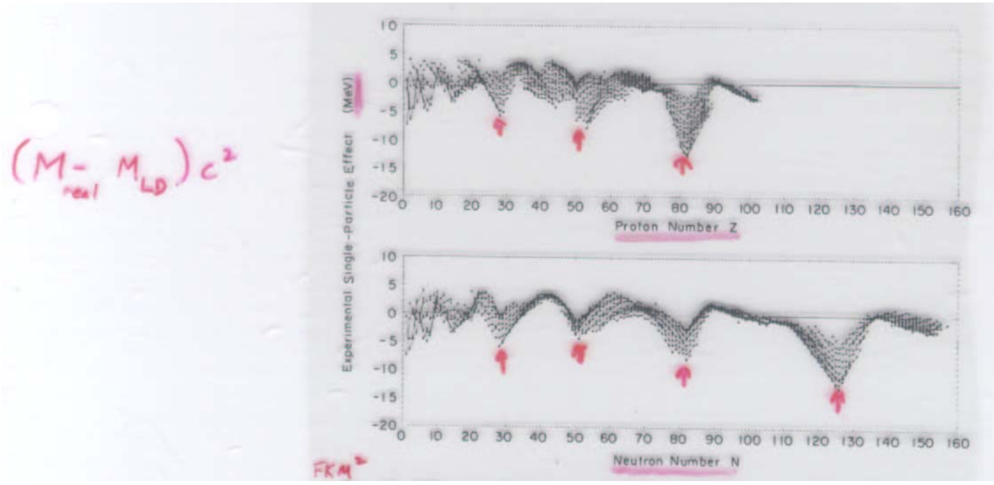


Fig. 8.2 Difference between measured masses and those predicted by a first-order liquid drop mass equation.

Interpretation

The liquid drop mass equation leads to the following general principles:

- Nuclei near ^{56}Fe are the most stable in Nature, which can be proved by differentiating Eq. 8.6 with respect to Z and A and setting the result to zero to obtain the minimum. This is the point at which the competition between surface energy losses and Coulomb energy losses is balanced, corresponding to the peak in the average binding energy curve in Fig.2.1. Therefore, **all nuclei are thermodynamically driven to ^{56}Fe.**
- Nuclei with **low A** lose binding energy primarily due to the surface energy term; i.e. as nuclei become smaller, a larger fraction of the nucleons are on the surface. By adding two light nuclei together in a **nuclear fusion** reaction, a more stable system is formed and **energy is released**. This is the principle upon which stars generate energy and is the basis for the nuclear fusion reactor, which someday may become a major source of commercial electricity (Sec.18).
- Nuclei with **high A** lose binding due to their large Coulomb energy. Thus, splitting a heavy nucleus in a **nuclear fission** reaction leads to more stable products and **energy is released**. Fission is the energy source for present-day nuclear reactors, also discussed in Sec. 18.
- The symmetry term favors N/Z = 1. For light nuclei this is observed up to ^{40}Ca. However, for heavier nuclei N/Z > 1. This trend is due to a balance with the Coulomb energy; i.e. larger A for a given Z lowers the Coulomb energy loss.
- Nuclei with even numbers of neutrons and protons gain additional binding relative to those with an odd nucleon.

Problem: Which is more stable, ^{124}Sn or ^{200}Hg?

Analysis of binding energy losses:

Surface energy – favors ^{200}Hg; however, for A > 56 surface energy loss is not a primary source of energy loss, since there is only a small change in $A^{2/3}$

.

Coulomb energy – favors ^{124}Sn; Z^2 is much larger for ^{200}Hg. Therefore, there is larger energy loss.

Symmetry energy – favors ^{124}Sn; N-Z =24 for ^{124}Sn versus 40 for ^{200}Hg. Therefore there is larger energy loss for ^{200}Hg.

Pairing energy – no effect; both are even-even nuclei.

Bottom line: **^{124}Sn is more stable** (closer to ^{56}Fe).

Line of Beta Stability

In the upper frame of Fig 2.1 a plot of N versus Z for the most stable nuclei shows the divergence from N = Z, as preferred by the symmetry term in the LD mass equation. To trace the evolution of this plot, it is useful to examine the prediction of the mass formula for individual **isobars A**, which are connected via beta decay (Section 3). Beta decay converts neutrons into protons or vice versa, as shown for the A = 75 chain of isobars.

$$^{75}_{30}\mathrm{Zn} \rightarrow {}^{75}_{31}\mathrm{Ga} \rightarrow {}^{75}_{32}\mathrm{Ge} \rightarrow \boxed{{}^{75}_{33}\mathrm{As}} \leftarrow {}^{75}_{34}\mathrm{Se} \leftarrow {}^{75}_{35}\mathrm{Br} \leftarrow {}^{75}_{36}\mathrm{Kr}$$

n → p   Beta-Stable Nucleus   p ← n

If A is held constant in the binding energy equation, then the mass defect for any isobar is given by

$$\left(\Delta {}^{A}_{Z}X \right) = d_1 Z^2 + d_2 Z\, d_3 + d + d_4 \delta \quad , \text{ where } d_i = f(C_i, A) \, , \quad \text{(Eq. 8.7)}$$

which is called the **isobaric mass formula**. This is the equation for a parabola and the minimum of the parabola corresponds to the most stable nucleus for a given isobar Z_A, which can be obtained by differentiating Eq. 8.7 and setting the result equal to zero. These minima define the 'valley of beta stability' corresponding to the topographic plot in Fig. 2.1 and shown more graphically in Fig. 8.3.

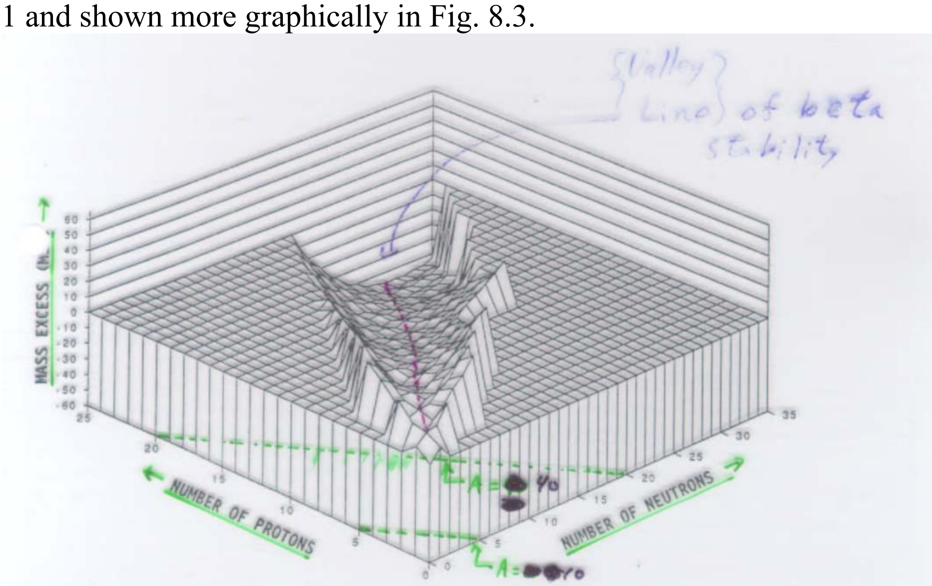


Fig. 8.3 A three-dimensional representation of the mass defect as a function of neutron and proton number showing the valley of nuclear stability.

Because of the pairing energy term in the mass equation, the solutions for even-A and odd-A nuclei yield different results. For odd-A nuclei the paring term is zero, leading to a single parabola for both e-o and o-e nuclei, as shown in fig. 8.4 for A = 75 and A = 157.

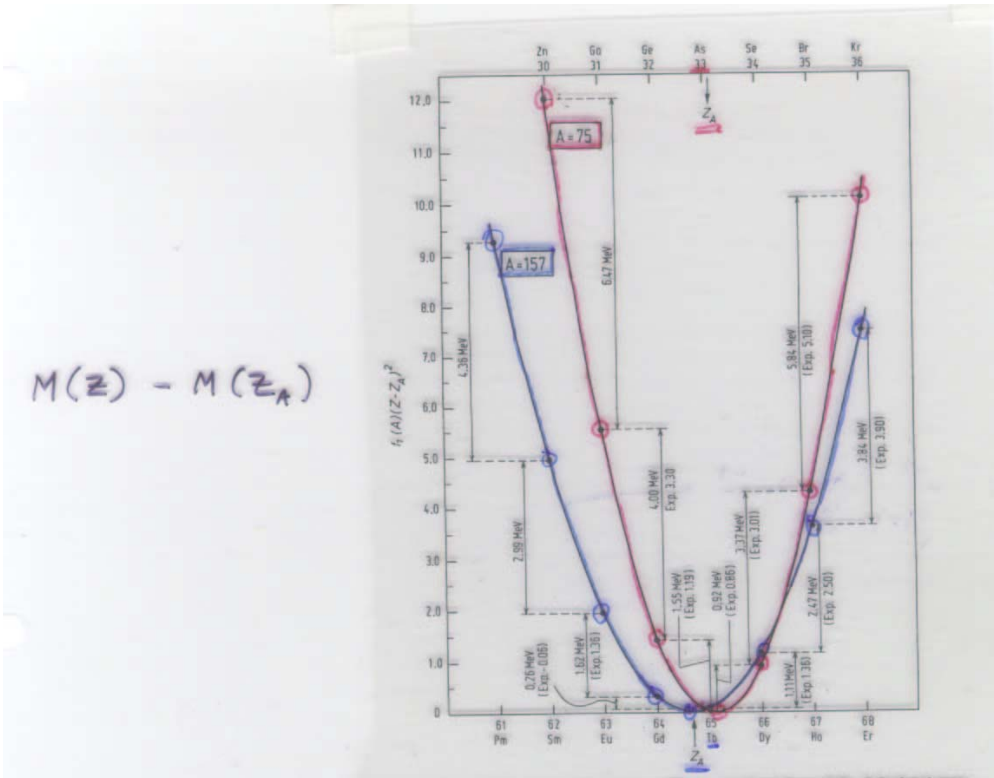


Fig. 8.4 Parabolic behavior of mass-energy differences for isobars with A = 75 and A = 157 nuclei, showing minima at Z_A, corresponding to the most stable isobar.

The important result for this case is that **there is only one beta-stable isotope for odd-A nuclei**.

For even A nuclei, the pairing term can be either +1 for e-e nuclei or -1 for o-o nuclei. This yields two parabolae as shown in fig 8.5. Since the beta decay of an e-e nucleus produces an o-o nucleus, the decay chain alternates between the two parabola. The upshot of this even-odd behavior is **that an even A nucleus can have up to three beta-stable isobars**, depending on the relative positions of the two parabola. In Fig. 8.5 it is observed for the A = 156 chain of beta decays, there are two stable isobars.

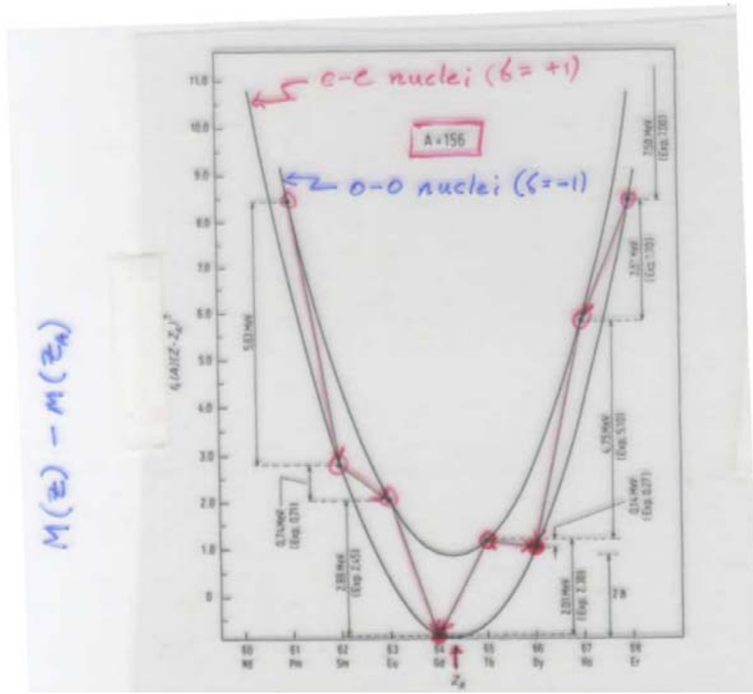


Fig. 8.5 Parabolic behavior of mass- energy differences for A = 15 nuclei, showing minima at most probable Z_{A}. The lower parabola represents e-e nuclei and the upper curve o-o nuclei.

The Z_A values translate into the black entries in the chart of the nuclides shown in Fig 8.6. Isobars lie along the diagonal from upper left to lower right. The color code represents the Q-values for beta decay.

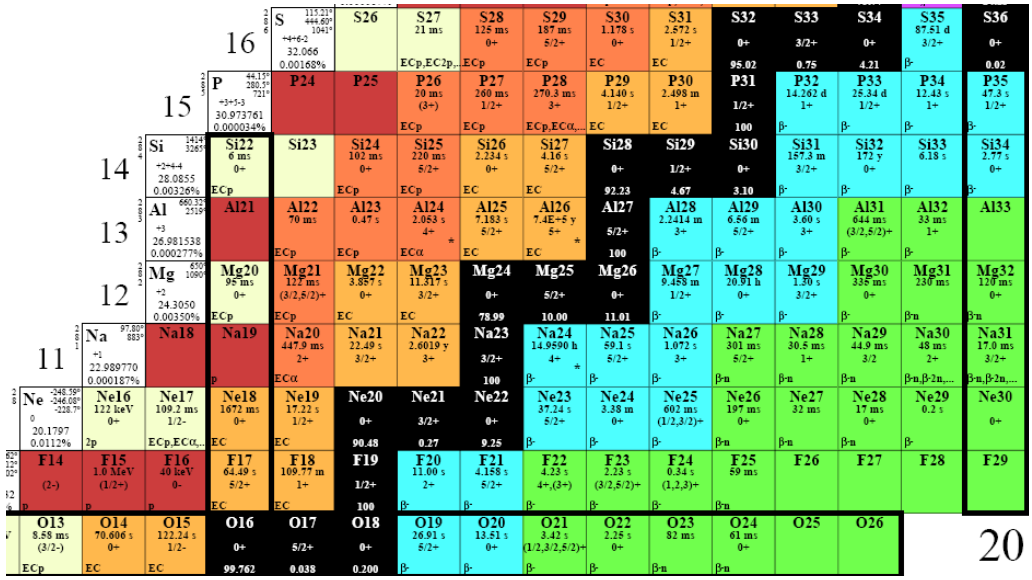
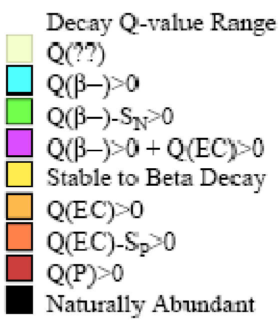


Fig. 5.6 Chart of the nuclides for A = 11 to A = 20



APPENDIX 8.1

Digression : How do we know that the mass of the neutron is **m(n) = 1.008 664 916 37(82) u ?**

More Precise Value of the Neutron Mass. The absolute wavelength of the gamma-ray produced in the reaction $n+p \rightarrow d+\gamma$ (2.2 MeV) was measured with a relative uncertainty of 2×10^{-7} using the NIST ILL GAMS4 crystal diffraction facility at the Institut Laue-Langevin in Grenoble, France. This wavelength measurement, expressed in energy units and corrected for recoil, is the binding energy of the neutron in deuterium. A previous crystal diffraction measurement of the deuteron binding energy has an uncertainty 5 times larger than this new result. The neutron mass follows directly from the reaction expressed in atomic mass units: $m(\mathrm{n}) = m(^2\mathrm{H}) - m(^1\mathrm{H}) + S(\mathrm{d})$ where S(d) is the separation energy of the neutron in deuterium. The uncertainties of the atomic mass difference, $m(^2\mathrm{H}) - m(^1\mathrm{H})$, and the new determination of S(d) are 0.71×10^{-9} u and 0.42×10^{-9} u, respectively, where u is unified atomic mass unit. The new, more precise value for the neutron mass, m(n) = 1.008 664 916 37(82) u, has an uncertainty which is ≈ 2.5 times smaller than the previous best value. [E. Kessler and M.S. Dewey (Div 846)]

Taken from **http://physics.nist.gov/TechAct.98/Div842/div842h.html**